Steinberg Groups for Jordan Pairs

Erhard Neher

University of Ottawa

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References:

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Universal central extensions

G abstract group

Definition (Schur)

- Central extension (=ce) $p\colon E \twoheadrightarrow G$ surjective group homomorphism, $\mathrm{Ker}(p)$ central
- universal central extension (= uce)
  1. $\hat{p}\colon \hat{G} \twoheadrightarrow G$ is ce, and
  2. for $p\colon E \twoheadrightarrow G$ ce there exists unique $f\colon \hat{G} \to E$, $p \circ f = \hat{p}$:

$$\begin{tikzcd} \hat{G} \arrow[rr, dotted, "\exists!\, f"] \arrow[dr, "\hat{p}"'] & & E \arrow[dl, "p"] \\ & G & \end{tikzcd}$$

- G centrally closed if $\mathrm{Id}\colon G \to G$ is uce, i.e., $E \underset{\dotsm}{\overset{\text{ce}}{\rightleftarrows}} G$

Schur: projective representations of finite groups

uce facts

Steinberg, Yale notes

G group: $\quad [g,h] = ghg^{-1}h^{-1}, \quad [G,G] = \langle [g,h] : g,h \in G\rangle$

1. uce unique, up to unique isomorphism
2. G has a uce $\iff$ $G = [G,G]$, i.e., G perfect
3. $\hat{p}\colon X \to G$ is a uce $\iff$
   (i) X is centrally closed (no condition on G!)
   (ii) $p\colon X \to G$ is a ce

Corollary of (3): Strategy to find uce

uce example

F field

$\mathrm{SL}_n(F) = \{X \in \mathrm{Mat}_n(F) : \det(X) = 1\} = \langle \mathrm{e}_{ij}(a) : 1 \le i \ne j \le n, a \in F \rangle$

$\mathrm{e}_{ij}(a) = \mathbf{1}_n + aE_{ij}$

linear Steinberg group $\mathrm{St}_n(F)$ presented by

generators $\mathrm{x}_{ij}(a)$, $1 \le i \ne j \le n$, $a \in F$,

relations $(a, b \in F)$

$$\begin{aligned} \mathrm{x}_{ij}(a)\, \mathrm{x}_{ij}(b) &= \mathrm{x}_{ij}(a+b) \\ [\mathrm{x}_{ij}(a), \mathrm{x}_{kl}(b)] &= 1 \quad \text{if } j \ne k \text{ and } l \ne i, \\ [\mathrm{x}_{ij}(a), \mathrm{x}_{jl}(b)] &= \mathrm{x}_{il}(ab) \quad \text{if } i, j, l \ne. \end{aligned}$$

Theorem (Steinberg 1962)

If $n \ge 4$, then $\mathrm{St}_n(F) \to \mathrm{SL}_n(F), \mathrm{x}_{ij}(a) \mapsto \mathrm{e}_{ij}(a)$ is a uce.

In general $\mathrm{St}_n(F) \to \mathrm{SL}_n(F)$ is not an isomorphism!

Generalizations

Recall Steinberg's Theorem: $\mathrm{St}_n(F) \to \mathrm{SL}_n(F)$, $n \geq 4$, is a uce

Generalizations

This theorem holds grosso modo in more generality:

- (Steinberg 1962) replace $\mathrm{SL}_n(F)$ by any Chevalley group, rephrase relations in terms of root systems
- (Stein 1972) replace $\mathrm{SL}_n(F)$ by Chevalley groups over commutative rings
- (Deodhar 1978) replace $\mathrm{SL}_n(F)$ by F–points of a quasi-split algebraic group
- (Kervaire-Milnor-Steinberg 1967/1971) in $\mathrm{St}_n(F)$ replace F by any ring,
- (Bak 1981) elementary unitary groups: rings with involutions (form rings), types $\mathrm{B}, \mathrm{C}, \mathrm{D}$

Kervaire-Milnor-Steinberg Theorem

A ring, define $\mathrm{St}_n(A)$ by presentation of $\mathrm{St}_n(F)$, F field:
generators $\mathrm{x}_{ij}(a)$, $1 \le i \ne j \le n$, $a \in A$,
relations $(a, b \in A)$

$$\begin{aligned} \mathrm{x}_{ij}(a)\, \mathrm{x}_{ij}(b) &= \mathrm{x}_{ij}(a+b) \\ [\mathrm{x}_{ij}(a), \mathrm{x}_{kl}(b)] &= 1 \qquad \text{if } j \ne k \text{ and } l \ne i, \\ [\mathrm{x}_{ij}(a), \mathrm{x}_{jl}(b)] &= \mathrm{x}_{il}(ab) \quad \text{if } i, j, l \ne. \end{aligned}$$

$\mathrm{E}_n(A) = \langle \mathrm{e}_{ij}(a) = \mathbf{1}_n + aE_{ij}, a \in A, 1 \le i \ne j \le n \rangle$, elementary linear group

Recall $\hat{p}\colon X \to G$ is a uce $\iff$ (i) X is centrally closed ($=$ its own uce) and (ii) $\hat{p}\colon X \to G$ is a ce.

Theorem (KMS)

A arbitrary ring,

(a) $\mathrm{St}_n(A)$, $n \ge 5$, *is centrally closed.*

(b) *If* $\mathrm{St}_n(A) \to \mathrm{E}_n(A)$, $\mathrm{x}_{ij}(a) \mapsto \mathrm{e}_{ij}(a)$, *is a ce, then it is a uce.*
This is so in the "stable" case:

$$1 \to \mathrm{K}_2(A) \to \varinjlim \mathrm{St}_n(A) \to \varinjlim \mathrm{E}_n(A) \to 1$$

$\mathrm{St}_n(A) \to \mathrm{E}_n(A)$ is not a central extension in general

Main result of [Loos-N.]

Theorem

Let

(i) (R, R_1) *be a locally finite irreducible* 3*–graded root system of rank* ≥ 5*,*

(ii) V *a Jordan pair with a fully idempotent root grading* $\mathfrak{R}$*.*

Then

(a) *the Steinberg group* $\mathrm{St}(V, \mathfrak{R})$ *is centrally closed.*

(b) *If the canonical map* $\mathrm{St}(V, \mathfrak{R}) \twoheadrightarrow \mathrm{PE}(V)$ *(= projective elementary group of* V*) is a ce, then it is a uce. This is so if* $\operatorname{rank} R = \infty$

Some advantages:

1. unified approach to linear and unitary Steinberg groups
2. new techniques, less relations
3. covers all known results, except split groups of type G_2, F_4, E_8, applies to new groups.

holds more generally: some rank 4, replace "fully"

Locally finite root systems = lfrs

lfrs $= \varinjlim$ (finite root systems $+0$) with respect to embeddings

Equivalent definition (à la Bourbaki): replace "finite" by "locally finite": $R \subset X$ such that $|R \cap Y| < \infty$ for every finite-dimensional subspace of X.

Rank of $R = \dim X$

irreducible lfrs $= \varinjlim$ (irreducible finite root systems)

Equivalent: R is not a direct sum of two non-zero root systems

Examples: Infinite rank generalizations of classical root systems, e.g.,
$\mathrm{A}_I = \{\epsilon_i - \epsilon_j : i, j \in \hat{I}\}, \hat{I} = I \dot\cup \{\star\}$
$\mathrm{B}_I = \{\pm\epsilon_i : i \in I\} \cup \{\pm(\epsilon_i + \epsilon_j) : i, j \in I, i \neq j\} \cup \{\epsilon_i - \epsilon_j : i, j \in I\}$

Classification (Kaplansky-Kibler 1973/75) irreducible lfrs = finite irreducible root system or isomorphic to A_I, B_I, C_I, D_I, BC_I, $|I| = \infty$.

3-graded locally finite root systems

3-grading $= \mathbb{Z}$–grading of lfrs R with support ± 1, 0.

Precise definition: decomposition $R = R_1 \dot\cup R_0 \dot\cup R_{-1}$ satisfying

$$R_{-1} = -R_1, \quad (R_i + R_j) \cap R \subset R_{i+j}, \quad R_0 = (R_1 + R_{-1}) \cap R.$$

Notation (R, R_1)

Example 3–gradings of $R = \mathsf{A}_{n-1} = \{\epsilon_i - \epsilon_j : 1 \le i, j \le n\}$, $p + q = n$

$R_1 = \{\epsilon_1 - \epsilon_{p+j} : 1 \le i \le p,\ 1 \le j \le q\}$

Facts:

1. 3–gradings of R respect decomposition of R into irreducible components
2. An irreducible lfrs R has a 3-grading $\iff$ R is reduced ($\neq \mathsf{BC}_I$) and not of type G_2, F_4, E_8.
3. R finite irreducible: ω minuscule coweight, $R_1 = \{\alpha \in R : \omega(\alpha) = 1\}$; every 3-grading of R of this type.

We now know assumption (i) of main theorem:
(R, R_1) is a locally finite irreducible 3–graded root system of rank ≥ 5,

Jordan pairs

k commutative ring

Jordan pair over k: $V = (V^+, V^-)$ pair of k–modules together with maps

$$Q^\sigma \colon V^\sigma \times V^{-\sigma} \to V^\sigma, \quad (x, y) \mapsto Q^\sigma(x)y = Q_x y, \qquad (\sigma = \pm),$$

quadratic in x and linear in y, satisfying certain identities.

Linearize $Q(x)y$ in x gives $Q_{x,z}y = Q(x,z)y = Q_{x+z}y - Q_x y - Q_z y$,
define *Jordan triple product*

$$\{\cdots\} \colon V^\sigma \times V^{-\sigma} \times V^\sigma \to V^\sigma, \quad (x, y, z) \mapsto \{x\,y\,z\} = Q_{x,z}y.$$

so $\{x\,y\,x\} = 2Q_x y$.

Examples

1. (Subpair) V Jordan pair, $S = (S^+, S^-) \subset V$ such that $Q(S^\sigma)S^{-\sigma} \subset S^\sigma$,
2. A associative k–algebra, $V = (A, A)$, $Q_x y = xyx$, $\{x\,y\,z\} = xyz + zyx$,
3. Combine (1) and (2)

Jordan pair examples

Examples

A associative k–algebra, $\mathfrak{A} = \mathrm{Mat}_n(A)$, so $(\mathfrak{A}, \mathfrak{A})$ Jordan pair with $Q_x y = xyx$.
Subpair $\mathbb{M}_{pq}(A) = \big(\mathrm{Mat}_{pq}(A), \mathrm{Mat}_{qp}(A)\big)$,

$$\begin{pmatrix} 0 & \mathrm{Mat}_{pq}(A) \\ \mathrm{Mat}_{qp}(A) & 0 \end{pmatrix} \quad \subset \quad \mathrm{Mat}_{p+q}(A)$$

since $(p \times q) \cdot (q \times p) \cdot (p \times q) = (p \times q)$.

other subpairs: symmetric, hermitian, alternating matrices

Root graded Jordan pairs

V Jordan pair, (R, R_1) 3–graded locally finite root system

Grosso modo: (R, R_1)–grading = grading by $\operatorname{span}_{\mathbb{Z}} R \subset X$, support in $R_1 \cup R_{-1}$

(R, R_1)-grading of V is a decomposition $V^\sigma = \bigoplus_{\alpha \in R_1} V_\alpha^\sigma$, $\sigma = \pm$, satisfying (RG1) and (RG2):

$$Q(V_\alpha^\sigma)V_\beta^{-\sigma} \subset V_{2\alpha-\beta}^\sigma, \qquad \{V_\alpha^\sigma\, V_\beta^{-\sigma}\, V_\gamma^\sigma\} \subset V_{\alpha-\beta+\gamma}^\sigma, \tag{RG1}$$

$$\{V_\alpha^\sigma\, V_\beta^{-\sigma}\, V^\sigma\} = 0 \qquad \text{if } \alpha \perp \beta. \tag{RG2}$$

Notation $\mathfrak{R} = (V_\alpha)_{\alpha \in R_1}$

Root graded Jordan pairs II

Example (Idempotents)

V Jordan pair over ring k, $e = (e_+, e_-) \in V$ with $e = \big(Q_{e_+}(e_-),\ Q_{e_-}(e_+)\big)$

Peirce decomposition

$$V^\sigma = V_2^\sigma(e) \oplus V_1^\sigma(e) \oplus V_0^\sigma(e), \qquad \sigma = \pm,$$
$$V_i^\sigma(e) = \{x \in V^\sigma : \{e^\sigma\, e^{-\sigma}\, x\} = ix\}, \qquad i = 0, 1, 2 \quad (\text{if } 1/2 \in k).$$

The $V_i^\pm = V_i^\pm(e)$ satisfy

$$Q(V_i^\sigma)V_j^{-\sigma} \subset V_{2i-j}^\sigma, \qquad \{V_i^\sigma\, V_j^{-\sigma}\, V_l^\sigma\} \subset V_{i-j+l}^\sigma,$$
$$\{V_2^\sigma\, V_0^{-\sigma}\, V^\sigma\} = 0 = \{V_0^\sigma\, V_2^{-\sigma}\, V^\sigma\},$$

where $i, j, l \in \{0, 1, 2\}$, $V_m^\sigma = 0$ if $m \notin \{0, 1, 2\}$.

Root-grading by $R = \mathrm{C}_2 = \{\pm\epsilon_i \pm \epsilon_j : i, j \in \{0, 1\}\}$, $R_1 = \{\epsilon_i + \epsilon_j : i, j \in I\}$,

$$V_\alpha^\sigma = V_{i+j}^\sigma(e), \qquad (\alpha = \epsilon_i + \epsilon_j \in R_1)$$

Fully idempotent root gradings

V Jordan pair, root grading $\mathfrak{R} = (V_\alpha)_{\alpha \in R_1}$ of type (R, R_1)

Recall $(\alpha, \beta \in R)$: $\langle \alpha, \beta^\vee \rangle = \beta^\vee(\alpha) \in \mathbb{Z}$, $\quad \alpha, \beta \in R_1$: $\langle \alpha, \beta^\vee \rangle \in \{0, 1, 2\}$,

Fully idempotent root grading $\mathfrak{R}$: every V_α, $\alpha \in R_1$, contains idempotent e_α such that for all $\beta \in R_1$

$$V_\beta = \bigcap_{\alpha \in R_1} V_{\langle \beta, \alpha^\vee \rangle}(e_\alpha)$$

Classification: N 1987

Example

A associative k–algebra, $\mathbb{M}_{pq}(A) = \big(\mathrm{Mat}_{pq}(A), \mathrm{Mat}_{qp}(A)\big)$, $Q_x y = xyx$,
$V^+ = \mathrm{Mat}_{pq}(A) = \bigoplus_{1 \le i \le p,\, 1 \le j \le q} A\, E_{ij}$, E_{ij} = matrix units
$R = \mathrm{A}_{p+q-1}$, $R_1 = \{\epsilon_i - \epsilon_{p+j} : 1 \le i \le p, 1 \le j \le q\}$
$\mathfrak{R} = (V_\alpha)_{\alpha \in R_1}$, $\quad V_{\epsilon_i - \epsilon_{p+j}} = (A\, E_{ij},\, AE_{ji})$ $\quad$ fully idempotent root grading

Recall

We now know the assumptions of

Theorem (Loos-N)

Assume

(i) (R, R_1) *be a locally finite irreducible* 3*–graded root system of rank* ≥ 5,

(ii) V *a Jordan pair with a fully idempotent root grading* $\mathfrak{R}$.

Then

(a) *the Steinberg group* $\mathrm{St}(V, \mathfrak{R})$ *is centrally closed.*

Steinberg group $\mathrm{St}(V)$

Definition (Steinberg group $\mathrm{St}(V, \mathfrak{R})$)

(R, R_1) 3-graded root system,
V Jordan pair with root grading $\mathfrak{R} = (V_\alpha)_{\alpha \in R_1}$, not necessarily idempotent.
Steinberg group $\mathrm{St}(V, \mathfrak{R})$ defined by presentation:

- generators $\mathrm{x}_+(u)$, $\mathrm{x}_-(v)$, $(u, v) \in (V^+, V^-)$;
- relations

$$\mathrm{x}_\sigma(u + u') = \mathrm{x}_\sigma(u)\, \mathrm{x}_\sigma(u') \qquad \text{for } u, u' \in V^\sigma, \tag{St1}$$

$$[\mathrm{x}_+(u), \mathrm{x}_-(v)] = 1 \qquad \text{for } (u, v) \in V^+_\alpha \times V^-_\beta,\ \alpha \perp \beta, \tag{St2}$$

$$\begin{cases} [\mathrm{b}(u, v), \mathrm{x}_+(z)] = \mathrm{x}_+(-\{u\, v\, z\} + Q_u Q_v z), \\ [\mathrm{b}(u, v)^{-1}, \mathrm{x}_-(y)] = \mathrm{x}_-(-\{v\, u\, y\} + Q_v Q_u y) \end{cases} \tag{St3}$$

for all $(u, v) \in V^+_\alpha \times V^-_\beta$ with $\alpha \neq \beta$ and all $(z, y) \in V$.

where for $\alpha \neq \beta \in R_1$, $(u, v) \in V^+_\alpha \times V^-_\beta$ define Bergmann operators $\mathrm{b}(u, v)$ by

$$\mathrm{x}_+(u)\, \mathrm{x}_-(v) = \mathrm{x}_-(v + Q_v u)\, \mathrm{b}(u, v)\, \mathrm{x}_+(u + Q_u v)$$

Steinberg group example

$R = \mathsf{A}_{n-1}$, $n = p + q \geq 5$, $R_1 = \{\epsilon_i - \epsilon_{p+j} : 1 \leq i \leq p, 1 \leq j \leq q\}$

$V = \mathbb{M}_{pq}(A) = \big(\mathrm{Mat}_{pq}(A), \mathrm{Mat}_{qp}(A)\big)$ Jordan pair $\{u\,v\,z\} = uvz + zvu$

fully idempotent root grading $\mathfrak{R}$ with $V_{\epsilon_i - \epsilon_{p+j}} = (AE_{ij}, AE_{ji})$,

The *Steinberg group* $\mathrm{St}(V, \mathfrak{R})$ is the group presented by

- generators $\mathrm{x}_+(u)$, $u \in V^+$, and $\mathrm{x}_-(v)$, $v \in V^-$, and
- the relations

$$\mathrm{x}_\sigma(u + u') = \mathrm{x}_\sigma(u)\,\mathrm{x}_\sigma(u') \quad \text{for } u, u' \in V^\sigma, \tag{St1}$$

$$[\mathrm{x}_+(u), \mathrm{x}_-(v)] = 1 \quad \text{for } (u, v) \in V_\alpha^+ \times V_\beta^-,\ \alpha \perp \beta, \tag{St2}$$

$$[\,[\mathrm{x}_\sigma(u), \mathrm{x}_{-\sigma}(v)], \mathrm{x}_-(z)] = \mathrm{x}_\sigma(-\{u\,v\,z\}) \tag{St3}$$

$$\text{for } u_\alpha \in V_\alpha^\sigma,\ v \in V_\beta^{-\sigma},\ z \in V^\sigma \text{ with } \langle \alpha, \beta^\vee \rangle = 1 = \langle \beta, \alpha^\vee \rangle.$$

Proposition

For $(V, \mathfrak{R})$ as above, $\mathrm{St}(V, \mathfrak{R}) \cong \mathrm{St}_n(A)$.
Hence $\mathrm{St}_n(A)$ is centrally closed (Part (a) of Kervaire-Milnor-Steinberg Theorem)

Tits-Kantor-Koecher algebra

Recall part (b) of Loos-N-Theorem:
"If the canonical map $\mathrm{St}(V,\mathfrak{R}) \to \mathrm{PE}(V)$ is a central extension, then it is a universal central extension. This is so, if rank $R = \infty$."

V Jordan pair over commutative ring k

Tits-Kantor-Koecher algebra of V is $\mathbb{Z}$–graded Lie k–algebra

$$
\begin{aligned}
\mathfrak{L}(V) &= \mathfrak{L}(V)_1 \oplus \mathfrak{L}(V)_0 \oplus \mathfrak{L}(V)_{-1}, \\
\mathfrak{L}(V)_0 &= k\zeta + \mathrm{span}_k\{\delta(x,y) : (x,y) \in V\}, \qquad \zeta = (\mathrm{Id}_{V^+}, \mathrm{Id}_{V^-}) \\
\delta(x,y) &= (D(x,y), -D(y,x)) \in \mathrm{End}(V^+) \times \mathrm{End}(V^-), \qquad D(x,y)z = \{x\,y\,z\}.
\end{aligned}
$$

Lie algebra product of $\mathfrak{L}(V)$ determined by

$$
\mathfrak{L}(V)_0 = \text{subalgebra of } \mathfrak{gl}(V^+) \times \mathfrak{gl}(V^-),
$$
$$
[V^\sigma, V^\sigma] = 0, \qquad [D,z] = D_\sigma(z), \qquad [x,y] = -\delta(x,y)
$$

Example $V = \mathbb{M}_{pq}(A)$

$$V = \mathbb{M}_{pq}(A) = \big(\operatorname{Mat}_{pq}(A), \operatorname{Mat}_{qp}(A)\big)$$

$$\operatorname{Mat}_{nn}(A) = \begin{pmatrix} \operatorname{Mat}_{pp}(A) & \operatorname{Mat}_{pq}(A) \\ \operatorname{Mat}_{qp}(A) & \operatorname{Mat}_{qq}(A) \end{pmatrix}$$

$$e_1 = \begin{pmatrix} \mathbf{1}_p & 0 \\ 0 & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_q \end{pmatrix}$$

$\operatorname{Mat}_{nn}(A)^{(-)}$ associated Lie algebra: $[x, y] = xy - yx$

$\mathfrak{e}$ = subalgebra of $\operatorname{Mat}_{nn}(A)^{(-)}$ generated by e_1, e_2 and V, $\mathfrak{z}(\mathfrak{e})$ = centre of $\mathfrak{e}$

$$\mathfrak{e}/\mathfrak{z}(\mathfrak{e}) \cong \mathfrak{L}(V)$$

Example

$A = K$ field of characteristic 0, $\mathfrak{e} = \mathfrak{gl}_n(K)$, $\mathfrak{L}(V) \cong \mathfrak{sl}_n(K)$

Projective elementary group $\mathrm{PE}(V)$

Recall: Jordan pair V, $Q_x y$

Tits-Kantor-Koecher $\mathfrak{L}(V) = \mathfrak{L}(V)_1 \oplus \mathfrak{L}(V)_0 \oplus \mathfrak{L}(V)_{-1}$

For $(x, y) \in V$: $(\operatorname{ad} x)^3 = 0$, so

$$\exp_+(x) = \mathrm{Id} + \operatorname{ad} x + \frac{1}{2}(\operatorname{ad} x)^2 = \begin{pmatrix} 1 & \operatorname{ad} x & Q_x \\ 0 & 1 & \operatorname{ad} x \\ 0 & 0 & 1 \end{pmatrix},$$

$$\exp_-(y) = \mathrm{Id} + \operatorname{ad} y + \frac{1}{2}(\operatorname{ad} y)^2 = \begin{pmatrix} 1 & 0 & 0 \\ \operatorname{ad} y & 1 & 0 \\ Q_y & \operatorname{ad} y & 1 \end{pmatrix}.$$

Define

$$\mathrm{PE}(V) = \big\langle \exp_+(x), \exp_-(y) : (x, y) \in V \big\rangle \subset \operatorname{Aut}\big(\mathfrak{L}(V)\big)$$

Example

V finite-dimension Jordan pair over $k = \bar{k}$ algebraically closed field:
$\mathrm{PE}(V)$ simple algebraic group of adjoint type and root system $\neq \mathrm{G}_2$, F_4, E_8, and conversely ... (scheme version available too).

Example $V = \mathbb{M}_{pq}(A) = \big(\mathrm{Mat}_{pq}(A), \mathrm{Mat}_{qp}(A)\big)$

Elementary group $\mathrm{E}(V)$ and projective elementary group $\mathrm{PE}(V)$ of V:

$$\mathrm{E}(V) = \left\langle \begin{pmatrix} \mathbf{1}_p & \mathrm{Mat}_{pq}(A) \\ 0 & \mathbf{1}_q \end{pmatrix} \cup \begin{pmatrix} \mathbf{1}_p & 0 \\ \mathrm{Mat}_{qp}(A) & \mathbf{1}_q \end{pmatrix} \right\rangle \subset \mathrm{GL}_n(A)$$

$$\mathrm{PE}(V) \cong \mathrm{E}(V) \big/ \mathrm{Z}\big(\mathrm{E}(V)\big)$$

Part (b) of Loos-N-Theorem:
If the canonical map $\mathrm{St}_n(A) \twoheadrightarrow \mathrm{PE}(V)$ is a cental extension, then it is a universal central extension. This is so in the the stable case.

Equivalent to part (b) of Kervaire-Milnor-Steinberg Theorem

Example

$A = K = \bar{K}$ algebraically closed field: $\mathrm{PE}(V) \cong \mathrm{PGL}_n(K)$.

Open problems: low ranks

J Jordan division algebra, e.g. $J = A$, A associative division algebra, $U_a b = aba$

$V = (J, J)$ Jordan pair with fully idempotent root grading of type
$R = \mathrm{A}_1$, $R_1 = \{\alpha\}$, $V_\alpha^\sigma = J$

Definition (Steinberg group St(J))

Notation of above. Steinberg group St(J) presented by

- generators $\mathrm{x}_+(u)$, $\mathrm{x}_-(v)$, $u, v \in J$;
  define Weyl $\mathrm{w}_b = \mathrm{x}_-(b^{-1})\,\mathrm{x}_+(b)\,\mathrm{x}_-(b^{-1})$ for $0 \neq b \in J$,
- relations

$$\mathrm{x}_\sigma(u + u') = \mathrm{x}_\sigma(u)\,\mathrm{x}_\sigma(u') \qquad \text{for } u, u' \in V^\sigma, \qquad \text{(StJ1)}$$

$$\mathrm{w}_b\,\mathrm{x}_-(a)\,\mathrm{w}_b^{-1} = \mathrm{x}_+\big(U(b)a\big) \quad (a \in J,\ 0 \neq b \in J.) \qquad \text{(StJ2)}$$

Question: Is St(J) centrally closed whenever $J \neq \mathbb{F}_q$ with $q \in \{2, 3, 4, 9\}$?

Answer by Steinberg: Yes, if A is a field.